# TD 8

# Cosmology

## 8.1 Robertson-Walker metrics

In this exercise we want to derive the Robertson-Walker metric based on two assumptions, that are:

- Spacetime can be sliced into hypersurfaces of constant time which are homogeneous and isotropic.
- The mean rest frame of the galaxies agree with this definition of simultaneity.

Therefore, we adopt comoving coordinates : each galaxy is idealized as having no random velocity, as it has a fixed set of coordinates $x^i$ (for $i = 1, 2, 3$). The time coordinate is $t$, the proper time for each galaxy.

1. Show that the spacetime we consider has a metric that can be written

$$\mathrm{d}s^2(t) = -\mathrm{d}t^2 + R^2(t) h_{ij} \mathrm{d}x^i \mathrm{d}x^j$$

   where $R(t)$ is a scale factor and $h_{ij}$ is a constant tensor. We will write $\mathrm{d}l^2 = h_{ij}\mathrm{d}x^i\mathrm{d}x^j$.

2. Using spherical symmetry around the origin, show that the spacial metric can be written

$$\mathrm{d}l^2 = e^{2\Lambda(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

   where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$.

3. Show that the Ricci scalar for this metric is

$$R = \frac{2}{r^2}\left(1 - \frac{\mathrm{d}}{\mathrm{d}r}\left(re^{-2\Lambda}\right)\right).$$

4. Deduce that the spacial metric is

$$\mathrm{d}l^2 = \frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\Omega^2 \tag{8.1}$$

   where $k$ is a constant.

5. Show that the full spacetime has metric

$$\boxed{\mathrm{d}s^2 = -\mathrm{d}t^2 + R^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\Omega^2\right]} \tag{8.2}$$

   where we can restrict $k \in \{1, 0, -1\}$.

6. We have shown that if a space that satisfies the two assumptions above exists, then there exists a constant $k$ such that its metric is (8.1). We still have to show that the converse is true, ie that this metric is isotropic and homogeneous for all $k \in \{1, 0, -1\}$.

   (a) Prove the statement in the case $k = 0$.

   (b) Consider now $k = 1$. We define a new coordinate $\chi(r)$ such that

   $$\mathrm{d}\chi^2 = \frac{\mathrm{d}r^2}{1 - r^2}\,.$$

   Write down $\mathrm{d}l^2$ using the coordinates $(\chi, \theta, \phi)$. Do you recognise this metric? Conclude.

   (c) Finally consider $k = -1$. Using a similar coordinate transformation, show that the metric of the spatial slices is proportional to $\mathrm{d}\chi^2 + \sinh^2\chi \mathrm{d}\Omega^2$. Is it possible to realise this 3-dimensional surface in a higher-dimensional Euclidean space? Show that $\mathrm{d}\chi^2 + \sinh^2\chi \mathrm{d}\Omega^2$ is the metric of a 3-dimensional hyperbola in Minkowski spacetime and conclude.

## 8.2 Lost in horizons

In this exercise, we adopt the following standard notations: $t_0$ refers to the date of "today", and for every quantity $X(t)$ that is a function of $t$, we define $X_0$ to be its value today : $X_0 = X(t_0)$. We use the normalized scale factor $a(t) = R(t)/R_0$ and the Hubble "constant"

$$H(t) = \frac{\dot{R}(t)}{R(t)} = \frac{\dot{a}(t)}{a(t)}\,. \tag{8.3}$$

The metric (8.2) can also be written

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + R^2(t)\left[\mathrm{d}\chi^2 + S_k^2(\chi)\mathrm{d}\Omega^2\right] \tag{8.4}$$

where $\chi$ is defined by $\mathrm{d}\chi^2 = \mathrm{d}r^2/(1-kr^2)$ and $S_k^2(\chi)$ is a function to be found. If we also introduce a new coordinate $\tau$ defined by $\mathrm{d}\tau = \frac{\mathrm{d}t}{R(t)}$, called the *conformal time*, the metric becomes

$$\mathrm{d}s^2 = R^2(t)\left[-\mathrm{d}t^2 + \mathrm{d}\chi^2 + S_k^2(\chi)\mathrm{d}\Omega^2\right] \tag{8.5}$$

We adjust the constants so that we sit at $r = \chi = 0$, and so that $R(t = 0) = 0$. The point $(t = 0, \chi = 0)$ will be called our *origin* (this is just a denomination, this point is in no way different from the points $(t = 0, \chi)$ for $\chi \neq 0$). Similarly the point $(t = +\infty, \chi = 0)$ will be called our *end*. Consider a galaxy at fixed $\chi$. Then we define the recession velocity (with respect to us) to be $\dot{R}(t)\chi$. We can then make the following definitions: a point in spacetime is said to be inside our

- *particle horizon* if it is possible for a particle to travel from our origin to this point;
- *event horizon* if it is possible for a particle to travel from this point to our end;
- *Hubble sphere* if its recession velocity is smaller than the speed of light.

1. Show that Friedmann's equation reads

   $$\dot{a}^2 = H_0^2\left(\frac{\Omega_M}{a} + \frac{\Omega_R}{a^2} + \Omega_\Lambda a^2 + (1 - \Omega_T)\right) \tag{8.6}$$

   where $\Omega_T = \Omega_M + \Omega_R + \Omega_\Lambda$, and where $M$ stands for the non-relativistic matter, $R$ for the radiation and $\Lambda$ for the cosmological constant.

2. Analyze and understand figure 8.1.

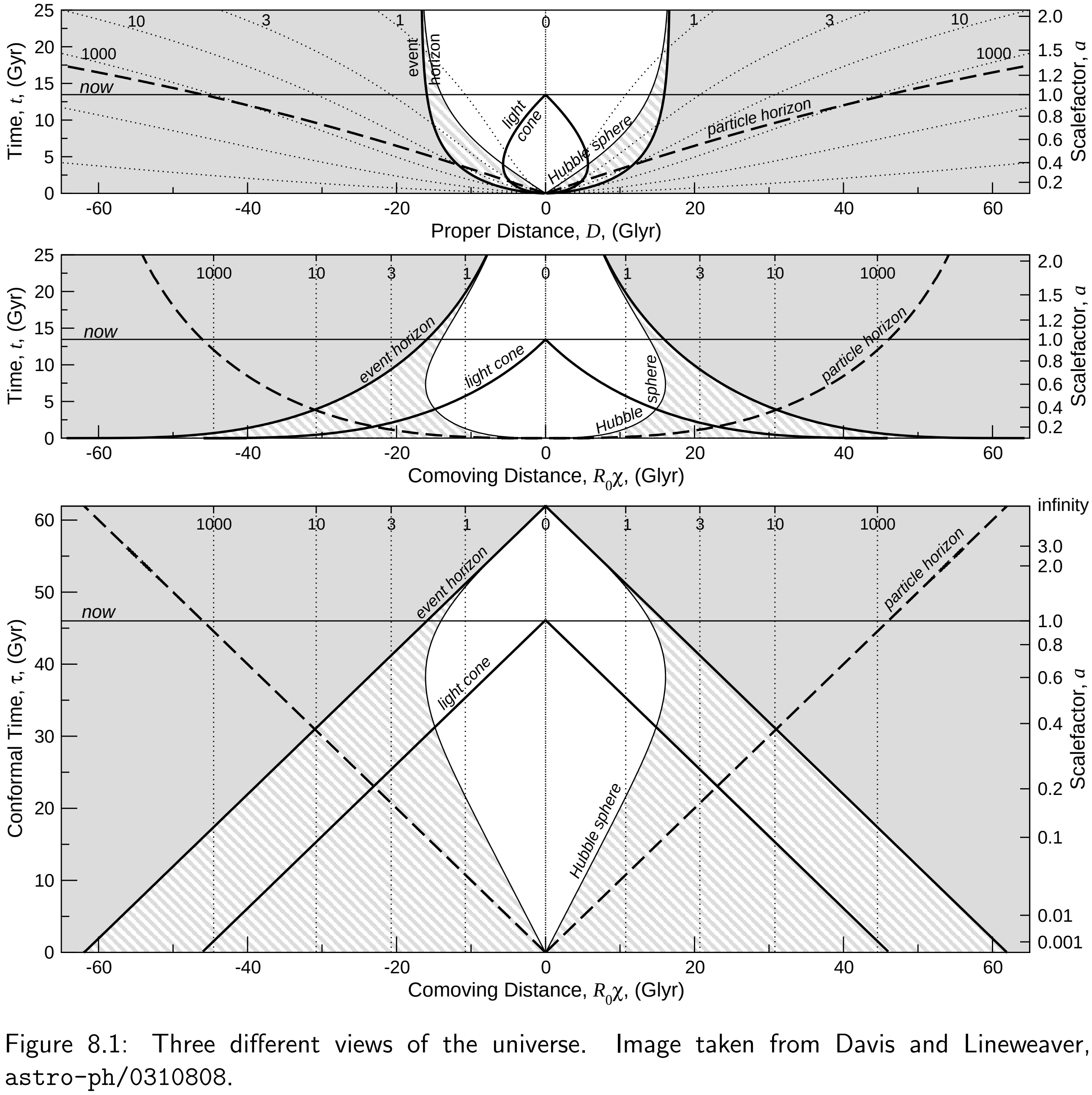


Figure 8.1: Three different views of the universe. Image taken from Davis and Lineweaver, `astro-ph/0310808`.

## 8.3 Scalar field in cosmology

We use a spacetime with metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

where $i, j = 1, 2, 3$. We remind Friedmann's equations for a perfect fluid with pressure $P$ and energy density $\rho$ :

$$\begin{aligned} 3\left(\frac{\dot{a}}{a}\right)^2 &= 8\pi G\rho \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3P) \end{aligned}$$

1. What is this spacetime? Is it flat?
2. We consider a scalar field $\phi(x^\mu)$. Its motion equation for an arbitrary metric $g_{\mu\nu}$ is

   $$g^{\mu\nu}D_\mu D_\nu\phi - \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0$$

   where $V(\phi)$ is the potential of the scalar field. Rewrite this equation using the scale factor $a(t)$ and its derivative.
3. The energy-momentum tensor associated to the scalar field is

   $$T_{\mu\nu} = D_\mu\phi D_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}D^\sigma\phi D_\sigma\phi + V(\phi)\right) .$$

   Check that it is conserved.
4. Show that the scalar field can be assimilated to a perfect fluid. Compute the pressure $P$ and the energy density $\rho$ of this fluid.
5. Let $w = P/\rho$. What is the behaviour of $\rho$ as a function of $a$ in the following approximations :
   (a) When the kinetic energy of the field is negligible in front of the potential energy ;
   (b) When the potential energy is negligible in front of the kinetic energy of the field.
6. Assuming that the scalar field dominates the universe, find the behaviour of $a(t)$ in those two situations.
7. Under which condition on $w$ does the expansion of the universe accelerate?